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# The Exigency of the Euclidean Parallel Postulate and the Pythagorean Theorem

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# The Exigency of the Euclidean Parallel Postulate and the Pythagorean Theorem

Jerry Lodder\*

## Notes to the Instructor

In this series of mini-projects we examine the use of the Euclidean parallel postulate (the fifth postulate of Book I of Euclid's *Elements* [1]) in the proof of the Pythagorean Theorem, essential today for computing the distance between two points in Euclidean space. The proof of the Pythagorean Theorem found in *The Elements* relies on the literal construction of squares on the sides of a right triangle. The construction of a square itself is contingent upon the Euclidean parallel postulate, as are many properties of parallelograms used in results leading to the Pythagorean Theorem. These properties follow from a basic result about parallel lines, namely the converse of the alternate interior angle theorem, which states that "A straight line falling on parallel straight lines makes the alternate angles equal to one another," [1, I.29], a result that is logically equivalent to the Euclidean parallel postulate. With so many results relying on the parallel postulate, how could it fail? What would replace the Pythagorean Theorem and the modern distance formula between two points?

Let us admit then to the exigency of the Euclidean parallel postulate, and examine the proof of the Pythagorean Theorem in detail. Euclid's writing is a model of deductive reasoning, with the logical dependencies of one proposition on previous propositions or postulates clearly stated. A careful reader can easily discern which results depend on the fifth postulate (the parallel postulate). Aside from a rigorous logical structure and the development of basic geometric results, why is the Pythagorean Theorem true? An understanding of the ancient Greek view of area as used in Book I of *The Elements* is essential to answer this question. At the heart of the Pythagorean Theorem is an area argument. The sum of the areas of the squares on the two legs of a right triangle is the area of the square on the hypotenuse. Euclid's view of area is very different from the modern point of view. First of all, no algebraic expressions for the area of even the simplest geometric figures were in use. Secondly, Euclid avoids numerical values, so a verbal expression of the area of figure in terms of a number is not given. Instead Euclid proves results stating when certain figures (parallelograms or triangles) have the same area. Parallelograms on the same base and contained between the same parallel lines have the same area. Likewise, triangles on the same base and

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contained between the same parallel lines have the same area. Not surprisingly, these area results depend on the Euclidean parallel postulate as well. The reader will recognize that the perpendicular distance between the two parallel lines containing the figure serves as the height of the figure, although this point of view is a bit anachronistic, and, in fact, counter-productive for an understanding of the Pythagorean Theorem. To understand Euclid on his own terms, draw literal squares on the sides of a right triangle. Then identify certain figures (triangles, parallelograms or squares) that are contained between two parallel lines. Drawing a line through a vertex of the triangle parallel to one of the sides of a square may be useful. The uniqueness of this parallel is, of course, equivalent to Euclid's fifth postulate.

Many of the propositions in Book I of *The Elements* are used either directly or indirectly in the proof of the Pythagorean Theorem. This series of mini-projects contains six installments, with the last two "The Ancient Greek View of Area," and "The Pythagorean Theorem" being pivotal. Readers with a solid background in high school geometry could begin with these two. For those wishing more details about the constructions leading to the Pythagorean Theorem, and in particular results depending on the parallel postulate, there are four preparatory mini-projects: "Construction of Perpendiculars," "The Alternate Interior Angle Theorem," "The Converse of the Alternate Interior Angle Theorem," and "On the Construction of a Square." Each mini-project begins with a statement of purpose, briefly describing how the result fits into the proof of the Pythagorean Theorem. This is followed by the prerequisite material needed for the constructions in a particular mini-project. The prerequisite material may be used without proof, such as the side-side-side congruency theorem for triangles. Also, the common notions and postulates of Book I may be used throughout the mini-projects. Next follows the statement of the proposition (or theorem) from *The Elements* that forms the core of the mini-project. Instead of reproducing Euclid's polished proof of the result, the mini-project outlines the strategy of the proof, allowing the reader to discover the key logical connections. These proofs could form the material for class discussion or class presentations. This is followed by exercises or questions that could be assigned as homework problems. Each mini-project is designed to be covered in about two class sessions. Ideally, the instructor would ask the students to read a mini-project in advance and come to class with particular questions.

For quick reference, here are the common notions and postulates from Book I. The instructor may wish to hand these out to the class. This list also appears on a separate page for easy duplication, after the last mini-project.

#### Common Notions.

1. Things which are equal to the same thing are also equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

## Postulates.

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

For reference, here is the list of mini-projects along with the proposition from *The Elements* developed in each.

## List of Mini-Projects.

1. Construction of Perpendiculars.  
Proposition 11: To draw a straight line at right angles to a given straight line from a given point on it.
2. The Alternate Interior Angle Theorem.  
Proposition 27: If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to another.
3. The Converse of the Alternate Interior Angle Theorem.  
Proposition 29: A straight line falling on parallel straight lines makes the alternate angles equal to one another.
4. On the Construction of a Square.  
Proposition 46: On a given straight line to describe a square.
5. The Ancient Greek View of Area.  
Proposition 35: Parallelograms which are on the same base and in the same parallels are equal to one another.
6. The Pythagorean Theorem.  
Proposition 47: In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

## References

- [1] Euclid, *The Thirteen Books of Euclid's Elements*, translated by Sir Thomas Heath, Vol. 1, Dover Publication, New York, 1956.

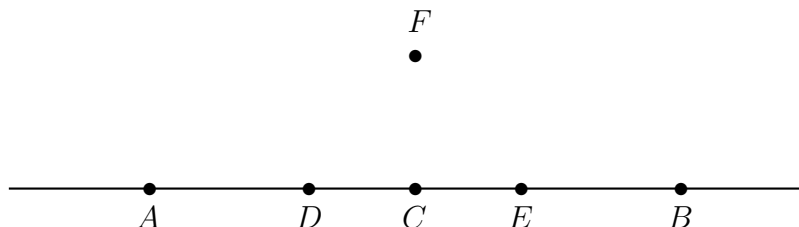
# 1 Construction of Perpendiculars

**Purpose:** The goal of this series of mini-projects is to gain insight into the proof of the Pythagorean Theorem found in Euclid's *Elements* [1]. This proof requires the literal construction of squares on the three sides of a right triangle. As a first step in the construction of a square, we examine the construction of a line perpendicular to a given line, found in Proposition 11 of the *Elements*.

**Prerequisite Material:** The *Elements* is written in a logically precise manner with one proposition building on the previous ones. The reader may use, without proof, the following results already proven in the *Elements*: segment duplication, construction of an equilateral triangle with a given side, the basic congruency theorems for triangles, such as side-side-side and side-angle-side. To establish the perpendicularity of two lines, it must be shown that the lines intersect at right angles. According to Euclid, "When a straight line set up on a straight line makes the adjacent angles equal to another, each of the equal angles is *right*" [1, p. 153].

**Proposition 11:** To draw a straight line at right angles to a given straight line from a given point on it.

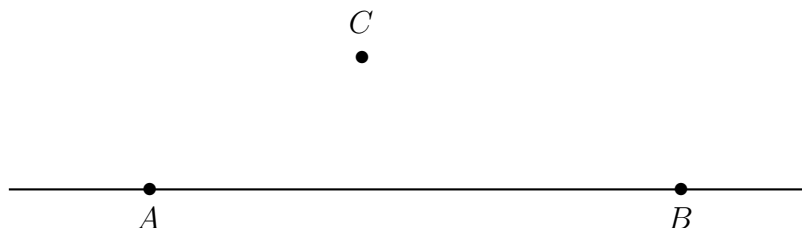
**Strategy of Proof:** Given line  $\overleftrightarrow{AB}$  and a point  $C$  on  $\overleftrightarrow{AB}$ , find some line through  $C$  perpendicular to  $\overleftrightarrow{AB}$ . To meet Euclid's definition of a right angle, we must find some line  $\mathcal{L}$  through point  $C$  so that the adjacent angles formed by  $\mathcal{L}$  and  $\overleftrightarrow{AB}$  (on one side of  $\overleftrightarrow{AB}$ ) are equal (or congruent). The equality of these angles will follow from the congruency of two triangles. But what two triangles? Consider another point  $D$  on  $\overleftrightarrow{AB}$ ,  $D \neq C$ . Construct a line segment  $CE$  on line  $\overleftrightarrow{AB}$  so that  $CE = DC$  in length. Then form an equilateral triangle  $\triangle DEF$  with one side being segment  $DE$ . Draw line segment  $FC$ . Can you find two congruent triangles? Can you prove that segment  $FC$  is perpendicular to line  $\overleftrightarrow{AB}$ ?



**Exercise 1.** In a step-by-step argument, prove that given line  $\overleftrightarrow{AB}$  and a point  $C$  on  $\overleftrightarrow{AB}$ , there is some line segment  $FC$  perpendicular to  $\overleftrightarrow{AB}$ . Be sure to offer a reason why each step holds. You may cite any previous proposition, definition, postulate or common notion from the *Elements* as a valid reason.

**Question 1.** Does the construction of a perpendicular line require the parallel postulate (Postulate 5 of the *Elements*)?

**Exercise 2.** In a step-by-step argument, prove that given line  $\overleftrightarrow{AB}$  and a point  $C$  **not** on  $\overleftrightarrow{AB}$ , there is a line segment through point  $C$  perpendicular to  $\overleftrightarrow{AB}$ . Be sure to offer a reason why each step holds. You may cite any previous proposition, definition, postulate or common notion from the *Elements* as a valid reason. You may also cite Proposition 9, angle bisection, or Proposition 10, segment bisection, as reasons.



**Question 2.** Does the construction in Exercise 2 rely on the parallel postulate?

## References

- [1] Euclid, *The Thirteen Books of Euclid's Elements*, translated by Sir Thomas Heath, Vol. 1, Dover Publication, New York, 1956.

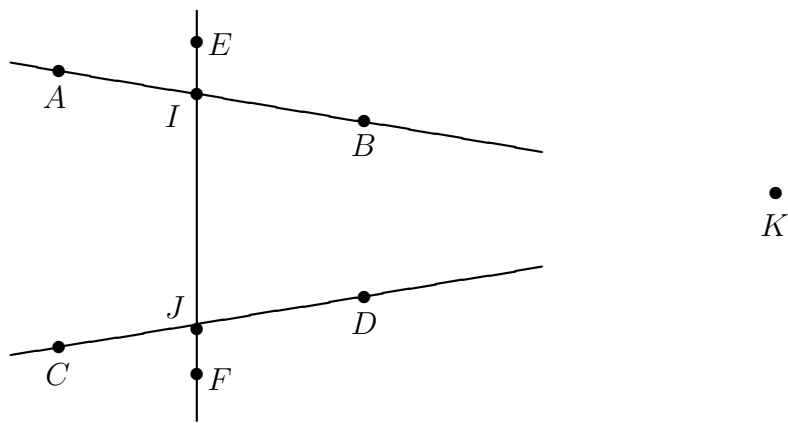
## 2 The Alternate Interior Angle Theorem

**Purpose:** Recall that the proof of the Pythagorean Theorem found in Euclid's *Elements* [1] requires the literal construction of squares on the three sides of a right triangle. In a previous mini-project we saw how to construct a line perpendicular to a given line, which is a first step in the construction of a square. Of course, opposite sides of a square must form parallel lines, a topic of this mini-project. Today, when one line intersects (falls on) two other lines, the three lines are said to form a transversal. In this mini-project we prove that if the alternate interior angles of a transversal are congruent, then two of the lines are parallel. We use Euclid's definition [1, p. 154] that "Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet another in either direction." We then prove Proposition 27 of the *Elements*, known today as the Alternate Interior Angle Theorem.

**Prerequisite Material:** The reader may use the postulates and common notions of the *Elements* as well as the exterior angle theorem for triangles, which states that the exterior angle of a triangle is greater than either of the opposite interior angles. Also, the vertical angle theorem and angle duplication may be used in the exercises.

**Proposition 27:** If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to another.

**Strategy of Proof:** Given lines  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$ , suppose that line  $\overleftrightarrow{EF}$  intersects  $\overleftrightarrow{AB}$  at point  $I$  and line  $\overleftrightarrow{EF}$  intersects line  $\overleftrightarrow{CD}$  at point  $J$ . Given that  $\angle AIJ \simeq \angle IJD$ , prove that line  $\overleftrightarrow{AB}$  is parallel to  $\overleftrightarrow{CD}$ . By Euclid's definition of parallel, we must show that lines  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$  do not meet. Using an indirect proof, assume that lines  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$  meet at some point  $K$  on the same side of  $\overleftrightarrow{EF}$  as point  $B$  (or point  $D$ ). Then consider triangle  $\triangle IKJ$ . Is  $\angle AIJ$  an exterior angle to triangle  $\triangle IKJ$ ? Can a contradiction be reached? What if lines  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$  intersect at some point  $L$  on the same side of  $\overleftrightarrow{EF}$  as point  $A$  (or point  $C$ ). What is your final conclusion?

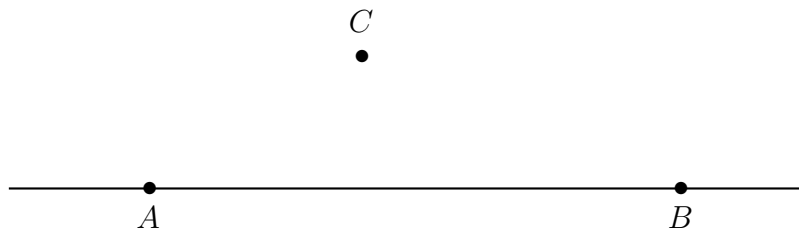


**Exercise 1.** Given lines  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$ , suppose that line  $\overleftrightarrow{EF}$  intersects  $\overleftrightarrow{AB}$  at point  $I$  and line  $\overleftrightarrow{EF}$  intersects line  $\overleftrightarrow{CD}$  at point  $J$ . Given that  $\angle AIJ \simeq \angle IJD$ , in a step-by-step argument, prove that line  $\overleftrightarrow{AB}$  is parallel to  $\overleftrightarrow{CD}$ . Use an indirect proof and provide the details of two cases, where the lines  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$  intersect on (i) the same side of  $\overleftrightarrow{EF}$  as point  $B$  and (ii) the same side of  $\overleftrightarrow{EF}$  as point  $A$ . Be sure to offer a reason why each step holds. You may cite any previous proposition, definition, postulate or common notion from the *Elements* as a valid reason.

**Question 1.** Does the proof of the Alternate Interior Angle Theorem (Exercise 1) require the parallel postulate (Postulate 5 of the *Elements*)?

**Exercise 2.** [The Corresponding Angle Theorem] Given lines  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$ , suppose that line  $\overleftrightarrow{EF}$  intersects  $\overleftrightarrow{AB}$  at point  $I$  and line  $\overleftrightarrow{EF}$  intersects line  $\overleftrightarrow{CD}$  at point  $J$ . Given that  $\angle EIB \simeq \angle IJD$ , in a step-by-step argument, prove that line  $\overleftrightarrow{AB}$  is parallel to  $\overleftrightarrow{CD}$ .

**Exercise 3.** In a step-by-step argument, prove that given line  $\overleftrightarrow{AB}$  and a point  $C$  not on  $\overleftrightarrow{AB}$ , there is some line through  $C$  parallel to line  $\overleftrightarrow{AB}$ . Be sure to offer a reason why each step holds.



**Question 2.** Does Exercise 3 (The Existence of Parallels) rely on the parallel postulate?

## References

- [1] Euclid, *The Thirteen Books of Euclid's Elements*, translated by Sir Thomas Heath, Vol. 1, Dover Publication, New York, 1956.



### 3 The Converse of the Alternate Interior Angle Theorem

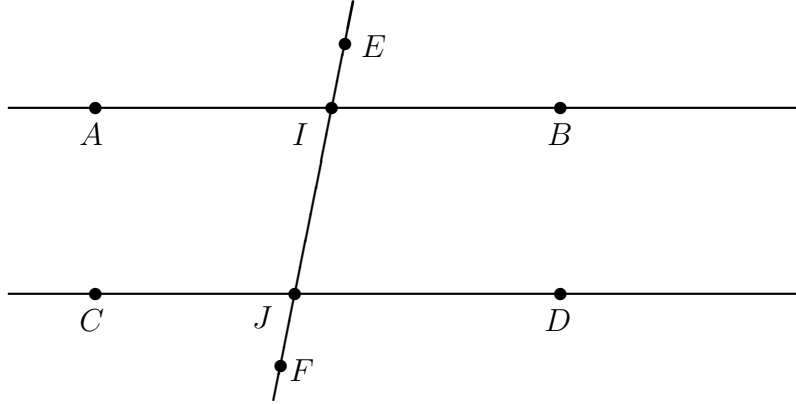
**Purpose:** We continue with a discussion of the propositions necessary for the construction of a square in Euclidean geometry, where Euclid’s fifth postulate, the so-called parallel postulate holds. A basic stepping stone in this direction is Proposition 29 of the *Elements* [1], the converse of the alternate interior angle theorem. First recall Euclid’s formulation of the fifth postulate [1, p. 155]: “if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.” Although wordy by today’s standards, the above formulation of the parallel postulate states the precise conditions needed for the proof of Proposition 29. In the sequel we will use a more streamlined version of the parallel postulate stated by the Scottish mathematician John Playfair (1748–1819) [2]: “Two straight lines cannot be drawn through the same point, parallel to the same straight line, without coinciding with one another.” Given a line  $\overleftrightarrow{AB}$  and a point  $C$  not on  $\overleftrightarrow{AB}$ , Playfair’s axiom asserts the existence of a unique line through  $C$  parallel to  $\overleftrightarrow{AB}$ . Recall that in the previous mini-project the existence of some line through  $C$  parallel to line  $\overleftrightarrow{AB}$  was proven without use of the parallel postulate. Thus, the main impact of Euclid’s fifth postulate is that the line through point  $C$  parallel to  $\overleftrightarrow{AB}$  is *unique*. In this mini-project we outline Euclid’s proof of Proposition 29 and discuss the logical equivalence between Euclid’s fifth postulate and Playfair’s postulate (stated above).

**Prerequisite Material:** The reader may use the postulates and common notions of the *Elements* as well as Proposition 13, which states that the angle sum around a line is  $180^\circ$ . Also, keep in mind Euclid’s version of the fifth postulate and Playfair’s version of the parallel postulate, written above. For the exercises, the vertical angle theorem and the existence of parallels may be used.

**Proposition 29 [abbreviated]:** A straight line falling on parallel straight lines makes the alternate angles equal to one another . . . .

**Strategy of Proof:** Given lines  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$ , suppose that line  $\overleftrightarrow{EF}$  intersects  $\overleftrightarrow{AB}$  at point  $I$  and line  $\overleftrightarrow{EF}$  intersects line  $\overleftrightarrow{CD}$  at point  $J$ . Given that  $\overleftrightarrow{AB}$  is parallel to  $\overleftrightarrow{CD}$ , prove that  $\angle AIJ \simeq \angle IJD$ . Using an indirect proof, assume that  $\angle AIJ$  is not congruent to  $\angle IJD$ . Then in measure, either (i)  $\angle AIJ > \angle IJD$  or (ii)  $\angle AIJ < \angle IJD$ . For case (i), consider  $\angle AIJ + \angle BIJ$ . If  $\angle AIJ > \angle IJD$ , what is true about  $\angle BIJ + \angle IJD$ ? Can Euclid’s fifth postulate now be applied? What contradiction is reached? Then consider case (ii), where  $\angle AIJ < \angle IJD$ .

**Exercise 1.** Given lines  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$ , suppose that line  $\overleftrightarrow{EF}$  intersects  $\overleftrightarrow{AB}$  at point  $I$  and line  $\overleftrightarrow{EF}$  intersects line  $\overleftrightarrow{CD}$  at point  $J$ . Given that line  $\overleftrightarrow{AB}$  is parallel to line  $\overleftrightarrow{CD}$ , in a step-by-step argument prove that  $\angle AIJ \simeq \angle IJD$ . Use an indirect proof and provide the details of two cases, where (i)  $\angle AIJ > \angle IJD$  and (ii)  $\angle AIJ < \angle IJD$ . Be sure to offer a reason why each step holds. You may cite any previous proposition, definition, postulate or



common notion from the *Elements* as a valid reason.

**Exercise 2.** [Converse of the Corresponding Angle Theorem] Given lines  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$ , suppose that line  $\overleftrightarrow{EF}$  intersects  $\overleftrightarrow{AB}$  at point  $I$  and line  $\overleftrightarrow{EF}$  intersects line  $\overleftrightarrow{CD}$  at point  $J$ . Given that line  $\overleftrightarrow{AB}$  is parallel to  $\overleftrightarrow{CD}$ , in a step-by-step argument, prove that  $\angle EIB \simeq \angle IJD$ . Be sure to offer a reason why each step holds.

**Exercise 3.** Given Playfair's parallel postulate, prove Euclid's fifth postulate. Be sure to offer a reason why each step holds.

**Exercise 4.** Given Euclid's fifth postulate, prove Playfair's parallel postulate. Be sure to offer a reason why each step holds.

## References

- [1] Euclid, *The Thirteen Books of Euclid's Elements*, translated by Sir Thomas Heath, Vol. 1, Dover Publication, New York, 1956.
- [2] Playfair, J. *Elements of Geometry: Containing the First Six Books of Euclid, with Two Books on the Geometry of Solids. To Which Are Added Elements of Plane and Spherical Trigonometry*, Bell & Bradfute and G. G. & J. Robinson, London, 1795.

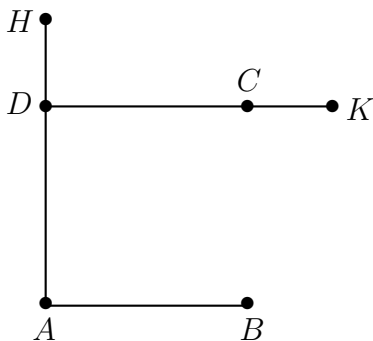
## 4 On the Construction of a Square

**Purpose:** In this mini-project we examine the construction of a square, needed later for the Pythagorean theorem. Once Euclid's fifth postulate (the parallel postulate) is accepted, then there are several possible strategies to prove the existence of a square. We outline one strategy which will have particular consequences in hyperbolic geometry, where unique parallels fail to exist.

**Prerequisite Material:** The reader may use all postulates and common notions of the *Elements* [1] as well as segment duplication, the existence of perpendicular lines, the alternate interior angle theorem and its converse. The triangle congruency theorems may also be used.

**Proposition 46:** On a given straight line to describe a square.

**Strategy of Proof:** Given line segment  $AB$ , we must find a quadrilateral (four-sided figure)  $ABCD$  that is a square, i.e., all four sides are congruent and all four angles are right angles. Construct a segment  $AH$  perpendicular to  $AB$ . Then on segment  $AH$  find a segment  $AD$  so that  $AD \simeq AB$ . From point  $D$ , construct a segment  $DK$  so that  $DK$  is perpendicular to  $AD$  and point  $K$  is on the same side of line  $\overleftrightarrow{AD}$  as point  $B$ . On segment  $DK$  construct segment  $DC$  so that  $DC \simeq AB$ . Connect points  $B$  and  $C$  with a line segment. Must figure  $ABCD$  form a square? We must show that both angles  $\angle DCB$  and  $\angle ABC$  are right angles, and we must show that segment  $BC$  is congruent to  $AB$  (or  $AD$ ). Are lines  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$  parallel? Is  $\angle DCA \simeq \angle BAC$ ? Why or why not? Is  $\triangle DCA \simeq \triangle BAC$ ? why or why not? What can be concluded about  $\angle ABC$ ? What can be concluded about segment  $BC$ ? Finally, find an argument showing that  $\angle BCD$  is a right angle.



**Exercise 1.** Given segment  $AB$ , in a step-by-step argument, prove the existence of a square  $ABCD$  on  $AB$ . Provide a reason why each step holds.

**Question 1.** Where is Exercise 1 is the parallel postulate being used?

**Exercise 2.** Given parallelogram  $ABCD$ , in a step-by-step argument, prove that

- (i)  $\triangle ABC \simeq \triangle CDA$ .
- (ii)  $AD \simeq BC$

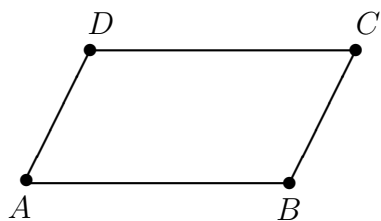
(iii)  $AB \simeq DC$

(iv)  $\angle DAB \simeq \angle BCD$

(v)  $\angle ADC \simeq \angle CBA$

(vi)  $\triangle DAB \simeq \triangle BCD$ .

Be sure to offer a reason why each step holds.



## References

- [1] Euclid, *The Thirteen Books of Euclid's Elements*, translated by Sir Thomas Heath, Vol. 1, Dover Publication, New York, 1956.

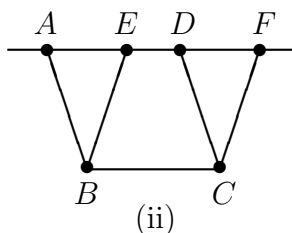
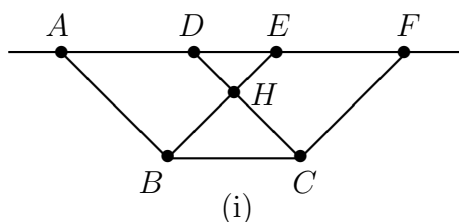
## 5 The Ancient Greek View of Area

**Purpose:** At the heart of the proof of the Pythagorean Theorem in Euclid's *Elements* [1] is an area argument. Given a right triangle, then the area of the square on the hypotenuse (the side opposite the right angle) is shown to be the sum of the area of the squares on the other two sides. How is area treated in the *Elements* given that algebra had not yet been developed in antiquity, precluding the use of any formula for area? Moreover, Euclid avoids the use of numbers in the *Elements* so an expression of area in terms of a numerical value is not given, even verbally. The key feature of figures (parallelograms or triangles) with equal area is that they are constructed on the same base and are contained between the same parallel lines. Once this view of area is understood, then the proof of the Pythagorean Theorem reduces to a geometric puzzle where certain figures (parallelograms or triangles) can be found on the same base and contained between the same parallel lines.

**Prerequisite Material:** The reader may use all postulates and common notions of the *Elements*. The triangle congruency theorems may be used as well as results about parallelograms asserting the congruency of opposite sides of a parallelogram as well as opposite angles of a parallelogram. Also, that a diagonal divides a parallelogram into two congruent triangles may be used without proof. Of course, congruent figures have equal area, a fact that Euclid uses liberally. In fact, Euclid writes that figures “are equal” in place of “have equal area.”

**Proposition 35:** Parallelograms which are on the same base and in the same parallels are equal to one another.

**Strategy of Proof:** Consider parallelograms  $ABCD$  and  $EBCF$ , both on base  $BC$  such that the opposite sides  $AD$  and  $EF$  are contained in the same line that is parallel to line  $\overleftrightarrow{BC}$ . Consider two cases, where (i) point  $D$  falls between points  $A$  and  $E$ , and (ii) where point  $E$  falls between points  $A$  and  $D$ . For case (i), argue why  $\triangle ABE \cong \triangle DCF$ . Suppose that segment  $BE$  intersects segment  $CD$  at point  $H$  in case (i). Then add the area of  $\triangle BCH$  to the area of  $\triangle ABE$  and compare this to the sum of the areas of  $\triangle BCH$  and  $\triangle DCF$ . Now subtract the area of  $\triangle HDE$  from both of these. What results? How much of the argument for case (i) can be applied to case (ii)?

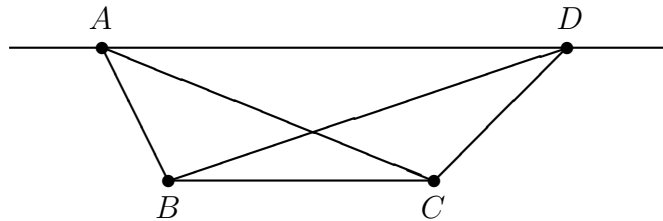


**Exercise 1.** In a step-by-step argument, prove the following. Consider parallelograms  $ABCD$  and  $EBCF$ , both on base  $BC$  such that the opposite sides  $AD$  and  $EF$  are contained in the same line that is parallel to line  $\overleftrightarrow{BC}$ . Then parallelograms  $ABCD$  and  $EBCF$  have

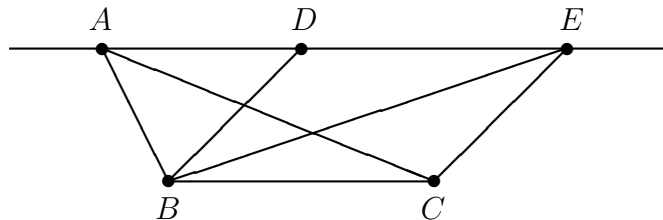
the same area. Be sure to provide the details of both cases above, and provide a reason why each step holds.

**Question 1.** Does the proof of Exercise 1 rely on the parallel postulate? How?

**Exercise 2.** Consider triangles  $\triangle ABC$  and  $\triangle DBC$ , both on base  $BC$  with point  $A$  different from point  $D$ . Suppose that line  $\overleftrightarrow{AD}$  is parallel to line  $\overleftrightarrow{BC}$ . In a step-by-step argument, prove that the area of  $\triangle ABC$  is equal to the area of  $\triangle DBC$ . Be sure to offer a reason why each step holds.



**Exercise 3.** Consider triangle  $\triangle ABC$  and parallelogram  $DBCE$ , both on base  $BC$  with points  $A$ ,  $D$  and  $E$  on the same line that is parallel to  $\overleftrightarrow{BC}$ . In a step-by-step argument, prove that the area of parallelogram  $DBCE$  is double the area of  $\triangle ABC$ . Be sure to offer a reason why each step holds.



## References

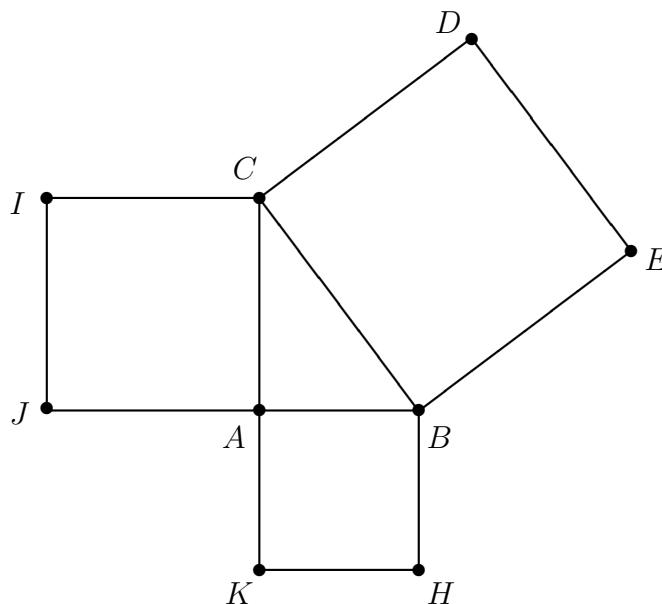
- [1] Euclid, *The Thirteen Books of Euclid's Elements*, translated by Sir Thomas Heath, Vol. 1, Dover Publication, New York, 1956.

## 6 The Pythagorean Theorem

**Purpose:** Book I of the *Elements* [1] ends with the proof of the Pythagorean Theorem and its converse. In the millennia following the *Elements*, this fundamental theorem has become the basis of the distance formula between two points in Euclidean geometry as well as many trigonometric identities such as  $\cos^2(\theta) + \sin^2(\theta) = 1$  for angles in Euclidean geometry. In this mini-project we examine the proof of the Pythagorean Theorem found in the *Elements*, based entirely on geometric squares and their area.

**Prerequisite Material:** The reader may use all postulates and common notions of the *Elements* as well as the triangle congruency theorems, the existence of a square on a given line segment, the existence of a line through a given point parallel to a given line, and results comparing the area of a triangle to a parallelogram. In particular, if a parallelogram and a triangle have the same base and be contained between the same parallel lines, then the area of the parallelogram is double that of the triangle. Also, if line segments  $AB$  and  $BC$  have angle measure  $180^\circ$  (two right angles), then points  $A$ ,  $B$  and  $C$  lie on the same line.

**Proposition 47:** In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.



**Strategy of Proof:** Given right triangle  $\triangle ABC$  with  $\angle CAB$  a right angle, first construct square  $BCDE$  on side  $BC$ . Then construct square  $ABHK$  on side  $AB$ . Then construct square  $ACIJ$  on side  $AC$ . We wish to show that the areas of squares  $ABHK$  and  $ACIJ$  add to the area of square  $BCDE$ . Are segments  $CA$  and  $AK$  on the same line? Why or why not? Are segments  $JA$  and  $AB$  on the same line? Why or why not? Consider lines  $\overleftrightarrow{CK}$  and

$\overleftrightarrow{BH}$  as a pair of parallel lines and form  $\triangle CBH$  with base  $BH$  contained between these two parallels. How does the area of this triangle compare with the area of square  $ABHK$ ? Find a new triangle with base  $BE$  congruent to  $\triangle CBH$ . Between what two parallels is this new triangle contained if one of the lines must be  $\overleftrightarrow{BE}$ ? How does the area of this new triangle compare to (part of) square  $BCDE$ . Now repeat the argument with square  $ACIJ$ . What is your final conclusion?

**Exercise 1.** In a step-by-step argument, prove that given right triangle  $\triangle ABC$  with  $\angle CAB$  a right angle, then the area of the square on side  $BC$  is the sum of the area of the squares on sides  $AB$  and  $AC$ . Be sure to offer a reason why each step holds.

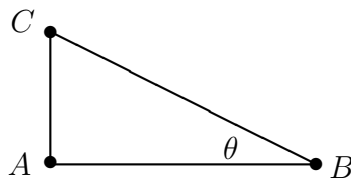
**Question 1.** Does the proof of Exercise 1 rely on the Euclidean parallel postulate? How?

**Exercise 2.** Suppose that  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  are points in the (Euclidean)  $xy$ -plane. Using modern algebraic techniques, show that the distance between  $P$  and  $Q$  is given by

$$\overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

**Exercise 3.** Given triangle  $\triangle ABC$  with right angle  $\angle CAB$ , let  $x = \overline{AB}$ ,  $y = \overline{AC}$  and  $r = \overline{BC}$ . Let  $\angle ABC$  have measure  $\theta$ . Then  $\cos \theta = \frac{x}{r}$  and  $\sin \theta = \frac{y}{r}$ . Using modern algebraic techniques, show that

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$



**Epilogue:** We have seen that the modern distance formula between two points is a direct consequence of the Pythagorean Theorem as is the basic trigonometric identity  $\cos^2(\theta) + \sin^2(\theta) = 1$ . The Pythagorean Theorem itself depends on the Euclidean parallel postulate. With such basic results relying on the parallel postulate, how could it fail? What would replace the distance formula? What would replace the trigonometric functions? We must wait for the pioneers of non-Euclidean or hyperbolic geometries.

## References

- [1] Euclid, *The Thirteen Books of Euclid's Elements*, translated by Sir Thomas Heath, Vol. 1, Dover Publication, New York, 1956.



# Euclid's Common Notions and Postulates

For easy reference, here are the common notions and postulates appearing in Book I of Euclid's *Elements*.

## Common Notions.

1. Things which are equal to the same thing are also equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

## Postulates.

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.